Stationary nonequilibrium ensembles for thermostated systems

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We explore the transient and the stationary behavior of proposed measures for chaotic dynamical systems, with particular reference to thermostated nonequilibrium systems. The Kawasaki distribution function formalism, which is based on the Liouville equation, is shown to give a good description of the transient measure. An orbital measure with a support based on unstable periodic orbits is shown to give the stationary measure. Also, we emphasize the importance of determining the set of observables by which a nonequilibrium measure can be defined. The stationarity of these methods is considered in both the periodic orbit theory and the Liouville picture for the change in the distribution function.

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I. INTRODUCTION

Progress in the statistical mechanics of nonequilibrium systems has been substantial over the past ten to fifteen years. This is particularly evident in the theory and simulation of perturbed equilibrium systems and the calculation of transport coefficients [1,2]. In these studies it is natural to consider an equilibrium (perhaps canonical) ensemble, to which differing reservoirs of mass, momentum, or energy are connected. The rate at which the mass, momentum, or energy is transported gives the desired transport coefficient. Conceptually this is a well-defined macroscopic system, but from a microscopic point of view it is difficult to handle both nonhomogeneous systems and the boundary between the system and reservoir. On the other hand, mechanical linear response theory for many-particle systems is relatively straightforward because the perturbing force acts explicitly on every particle in the system. This led to the first significant advance where it was realized that it was possible to transform a boundary driven thermal process into an equivalent homogeneous mechanical process, where an external field acts on each particle. Under the action of the field the dissipative heating is proportional to the square of the field, so going from linear to nonlinear systems requires the use of some form of heat reservoir to achieve a steady state (stationary) system. As before, this is difficult to achieve microscopically, so the solution (which we consider as the second major advance) was to transform the effect of a heat reservoir into an equivalent homogeneous mechanical process. To do this we need to define the temperature microscopically and then use some mechanical constraint mechanism to maintain this temperature to be constant (on either a microscopic or macroscopic time scale). The Gaussian isokinetic thermostat [1] keeps the total kinetic energy constant instantaneously (microscopic time scale), whereas the Nosé-Hoover thermostat [3] fixes the average total kinetic energy (macroscopic time scale). Applying the two transformations (boundary condition to field and boundary condition to thermostat) to unperturbed equilibrium ensembles, we obtain what we regard as generic nonequilibrium ensembles and it is these to which we restrict our considerations in this paper. It should be clear that not all nonequilibrium situations can be treated in the way suggested above and these ensembles are generic in the sense that we can apply them in many circumstances to the study of the transport of conserved quantities (mass, momentum, and energy).

Here we will consider the nonequilibrium ensemble obtained from the color conductivity algorithm [4] with a Gaussian isokinetic thermostat [5]. Choosing the system to be infinitely periodic, with two particles in the central (or simulation) cell, we obtain the simplest nontrivial system that behaves in a way that we recognize as macroscopically irreversible. That is, if we apply the perturbing external field in a particular direction we always observe a macroscopic current in the same direction (given a suitable sign convention for the current) and never in the opposite direction. A trivial coordinate transformation maps this color conductivity system to a nonequilibrium Lorentz gas [6,7]. The underlying microscopic dynamics for this model is a set of equations of motion that are time reversible, so that for every segment of phase space trajectory that produces a physical current (that is, one in the macroscopically observed direction) its time reverse also occurs and has exactly the opposite current. This produces one form of Löschmidt's paradox [8], which can be formulated as follows: how does irreversible behavior emerge from systems of particles with reversible equations of motion? In this case, if both the forward and time reverse trajectory exist, how does a nonzero current arise? For our particular situation we can now provide a dynamically based answer to those questions [9].

The phase space distribution function (by which we may define a probability measure in phase space) is well understood for systems that obey equilibrium statistical mechanics. The microcanonical, canonical, and grand-canonical ensembles are used regularly to model such systems. The description of nonequilibrium systems, on the other hand, is much less developed and generally very poorly understood. However, recent advances in the theory of dynamical systems are having a very significant impact in nonequilibrium statistical mechanics, given the

unifying power of their language, which does not differentiate between equilibrium and nonequilibrium systems. It is the aim of this paper to apply a technique from dynamical systems theory—periodic orbit theory—in the framework of nonequilibrium statistical mechanics. We begin by illustrating the problems with a pedagogical example: the generalized baker's map [10,11]. The features of this mapping help us to better understand some aspects of the dynamics of more physically interesting systems. Then, we make the connection between the Kawasaki formalism [1], a classical tool of nonequilibrium statistical mechanics, and periodic orbit theory in the special example of the nonequilibrium Lorentz gas. This allows us also to answer some questions concerning the Kawasaki formalism.

The present work is organized as follows. Section II introduces the baker's map and discusses the convergence of its dynamics to a stationary state. Section III is a brief introduction to periodic orbit theory, while Sec. IV applies the prescriptions of such a theory to the baker's map. Section V presents the Kawasaki formalism, which is then applied to the nonequilibrium Lorentz gas in Sec. VI. In Sec. VII the applicability of the Kawasaki formalism to a stationary state is discussed in terms of periodic orbits. Section VIII contains our concluding remarks, with some comment on the relevance of generalized dimensions for our problems.

II. THE DYNAMICAL EVOLUTION OF THE BAKER'S MAP

Quite clearly the study of realistic models is complicated enough to hide many interesting features that can be easily understood in terms of simple mappings. Therefore, in this section we investigate the processes occurring in the transient evolution of a simple dynamical system—the generalized baker's map [10,11]—that lead to the final stationary distribution. In a generic chaotic dynamical system, a given initial distribution in phase space is subjected to a process of stretching, contraction, and folding, which changes the distribution at every time step, until a final stationary state is achieved. If the system is dissipative, the contraction is stronger than the stretching and the final distribution has a support that is smaller than the initial support, while the densities of the associated measures also change. (The support of a measure is the complement of the largest open set of vanishing measure.) All of those effects are found in the generalized baker's map, of which we now study a particular case. Let $[0,1]\times[0,1]$ be the phase space and denote a phase point by (x,y). The action of the map is given by

$$\begin{aligned} x_{n+1}&=\Lambda_2 x_n, \\ y_{n+1}&=\Lambda_1 y_n, \end{aligned} \quad 0\le y_n<\tfrac{1}{2}\ ; \qquad (1)$$

$$\begin{aligned} x_{n+1}&=\Lambda_2 x_n+\tfrac{1}{2}, \\ y_{n+1}&=\Lambda_1 y_n-1, \end{aligned} \quad \tfrac{1}{2}\le y_n<1\ ,$$

where $\Lambda_1=2$ and $\Lambda_2\le\frac{1}{2}$ are the expansion and contraction factors for the map. If $\Lambda_2=\frac{1}{2}$ the map preserves the uniform distribution (Lebesgue measure) on the square and this fact is evident on all possible length scales. This is typical of equilibrium processes, like the Lorentz gas in the absence of external fields [12], or of nonequilibrium processes that take place in isolated containers, like the adiabatic expansion of a gas [13]. This distinguishing feature here is that the microscopic evolution is purely Hamiltonian. However, we can find an analogy between the baker's map and dissipative systems, which do not preserve the Lebesgue measure, by leaving $\Lambda_1=2$ and reducing Λ_2 in our baker's map. Then, smaller values of Λ_2 would correspond to the Lorentz gas with increasing values of the field [7]. In this section we study the transient behavior of the dissipative baker's map, with the aim of obtaining the stationary behavior in the $n\to\infty$ limit, and we treat in detail the particular case $\Lambda_2=\frac{1}{3}$.

In the first iteration, the uniform measure on the square evolves in the following way. We see, in Fig. 1(a),

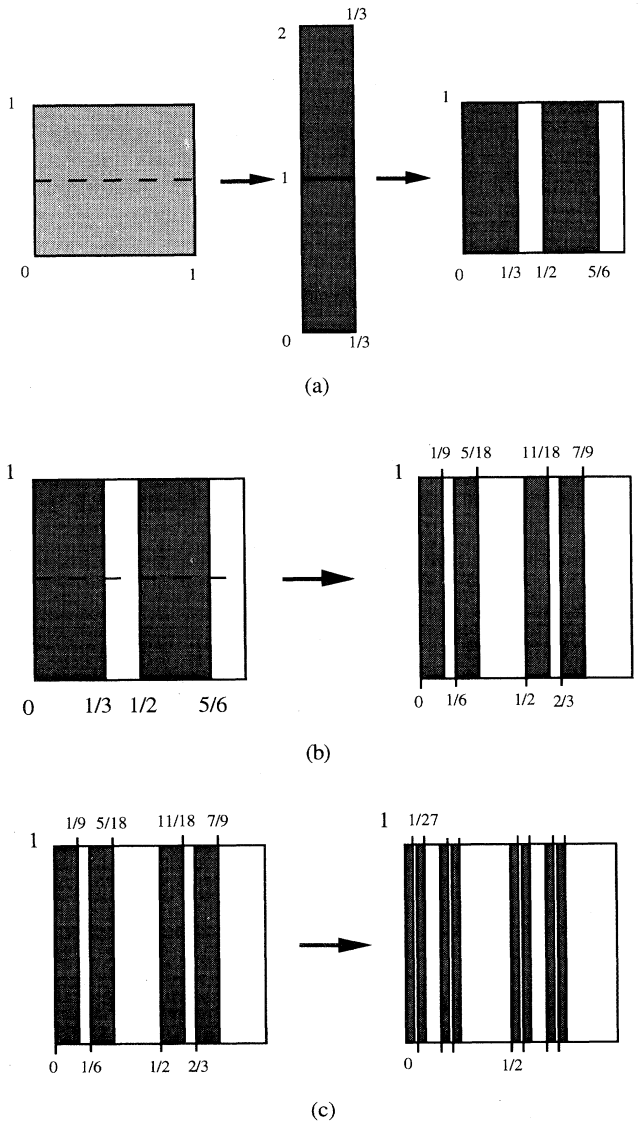

FIG. 1. (a) First iteration of the baker's map acting on a uniform measure on the square. (b) Second iteration. (c) Third iteration. At each step the number of rectangles increases by a factor of 2 and the density on each increases.

that the initial square is stretched and then cut and the two pieces are mapped back into the initial square. There are now two areas with vanishing measure and the support of the measure has shrunk, while the density on it has increased, in such a way that the total mass of the measure on the square is still one. In the second iteration [Fig. 1(b)] two rectangles become four rectangles and again the support is diminished, while the density is increased to maintain the normalization of the measure.

These changes continue forever and in the process we might expect to approach a stationary distribution, in accord with our expectations for physical dissipative systems. We will return to the problem of stationary measures later, but at present we restrict our considerations to the *transient measure.* The evolution of the transient measure can be followed by simply noting the position of the left boundary of each rectangle in the support (which we shall refer to as the *left line),* the widths of these rectangles, and the uniform density of the measure in them. In Table I we summarize the results for the first few steps of the baker's map where the initial distribution is uniform on the unit square. Note that, starting from the uniform distribution in phase space, the dynamics evolve the distribution into one that is nonuniform in the x direction, but still uniform in the y direction. Indeed, in one iteration of the map every existing rectangle has its rightmost and the central sixth removed, producing two new rectangles whose combined area is equal to two-thirds of the original one. At the same time we observe that the left line of the original rectangle remains and a new left line is created.

Considering the baker's map as a system of statistical mechanical interest, we may ask how the relevant observables evolve in time during the transient. This amounts to computing the averages of those functions of phase with the transient measures produced by iterating the map, starting from an initial ensemble (uniform in our example). For instance, we can calculate the moments of the transient distributions and it can be shown that after M iterations the average value of x is given by

$$\langle x \rangle_M = \frac{3}{8}\left[1+\frac{1}{3^{M+1}}\right], \tag{2}$$

for which the limit as $M\to\infty$ is well defined and equals $\langle x \rangle_\infty = \frac{3}{8}$. Equation (2) depends upon the initial distribution being uniform. It is easy to see that if we take a different initial measure, e.g., one that is uniform for $0 \le y \le \frac{1}{2}$ and zero otherwise, the limiting average is unchanged, but the approach to the limit is different. The next moment of the distribution can be shown to converge to $\langle x^2 \rangle_\infty = \frac{27}{128}$.

TABLE I. Structure of the transient measure for the baker's map.

Step	Left lines	Width	Density
0	0	1	1
1	$0, \frac{1}{2}$	$\frac{1}{3}$	$\frac{3}{2}$
2	$0, \frac{1}{6}, \frac{1}{2}, \frac{2}{3}$	$\frac{1}{9}$	$\frac{9}{4}$
3	$0, \frac{1}{18}, \frac{1}{6}, \frac{2}{9}, \frac{1}{2}, \frac{5}{9}, \frac{2}{3}, \frac{13}{18}$	$\frac{1}{27}$	$\frac{27}{8}$
⋮			
M		$\frac{1}{3^M}$	$(\frac{3}{2})^M$

For this system, we can also define and calculate the Gibbs entropy after M iterations to be

$$S_G^{(M)} = -\int d\Gamma\, f(\Gamma,M)\ln f(\Gamma,M) = -M\ln(\tfrac{3}{2}), \tag{3}$$

where $f(\Gamma,M)=(\frac{3}{2})^M$ on its support and zero elsewhere. We see that $S_G^{(M)}$ diverges linearly with M: $S_G^{(M)} \xrightarrow{M\to\infty} -\infty$. Similarly, if the Boltzmann entropy is defined to be the logarithm of the volume of phase space accessible to a particular macrostate and we assume that the fraction of phase space covered by the rectangles of nonvanishing measure corresponds to a single macrostate, we get that the Boltzmann entropy behaves like

$$S_B^{(M)} = \ln\left[\sum_{i=1}^{2^M} (\tfrac{1}{3})^M\right] = -M\ln(\tfrac{3}{2}). \tag{4}$$

This is precisely the result we obtained for the Gibbs entropy and it appears that we have an example of a nonequilibrium, dissipative system where the two entropies are the same at every state of the time evolution. The values calculated here are those obtained by iterating an initial uniform measure. Other choices for the initial distribution would, in general, lead to different values for finite values of M.

As the evolution proceeds, the system is expected to approach a stationary state, thus it seems natural to consider what we would call the stationary measure for our map, by looking at the behavior of the transient measures in the limit as $M\to\infty$. As we have seen, each application of the baker's map deletes two strips from each rectangle and doubles the number of rectangles, endowing the phase space with finer and finer structure. This process continues indefinitely in such a way that the distribution remains normalized and uniform on the current rectangles. Therefore, every point of the phase space that does not belong to a left line of a rectangle finds itself inside a region of zero measure, after sufficiently many iterations, and it would appear that the stationary distribution has to be zero everywhere, except on the left lines, where it should be uniform. However, any measure with a density supported on a finite number of left lines immediately appears not to be stationary, as its mass is redistributed over twice as many left lines with each iteration, thus changing its support. Similarly, a uniform measure supported on all the infinitely many left lines would appear to be nonstationary and unnormalized. In fact, consider a distribution that assigns a mass μ uniformly to each left line. One application of the map moves the mass of a given left line onto two other left lines, so that the mass on each left line decreases at every iteration. This actually holds for all measures that have a density along the length of the left lines (i.e., those that are not singular in the y direction). In fact, the dynamics merely stretches the relevant distributions along the y direction, with the total mass of a measure, nonetheless, always preserved.

These considerations look puzzling from a physical

point of view and complement the observation that the behavior of the Gibbs and Boltzmann entropies suggest that the limit of the transient measures, as $M \to \infty$, may not even be well defined. Thus we may ask what it means to start on the stationary state. Do we have a way of constructing the stationary distribution by assigning a certain weight to all the reasonable (e.g., Borel) subsets of phase space?

A common way of avoiding this problem is to consider a coarse graining of the phase space. We might interpret this as some inherent limit of resolution. This leads to an observation of stationarity when the measure is changing on a length scale that is smaller than the graining length. This aspect is not different from the Lebesgue measure preserving case when we start with a different initial distribution. In fact, if we do not start with the uniform measure, we will only approximate it more and more closely as the number of iterations grows, but we will never actually achieve it, unless we look at the phase space in a nondetailed way. Thus the result of a coarse graining is to make the measure seem to be stationary, while leaving open the question of the arbitrariness of the graining.

However, there is a difference between the stationary distributions of Lebesgue measure preserving and nonpreserving maps. Indeed, in the case of the measure preserving baker's map, we could start with the uniform distribution in phase space, which would be preserved by the dynamics at all length scales. On the contrary, for the dissipative map we have considered, all distributions appear not to be invariant under the application of the map, unless a coarse graining is done. We argue that this difference is what distinguishes nonequilibrium stationary states from equilibrium ones, on the fine-grained level.

The question is then at what level can we speak of stationary states for nonequilibrium systems if the phase space distribution functions are not invariants. In physical terms, the problem is that the phase space distribution is not a proper observable for the system and so there is no way of testing it directly. In mathematical terms, this is faced by studying measures, and their evolutions, by furnishing the space to which they belong with the appropriate topologies. In this way, the properties of measures defined through a limiting procedure are not tested directly but through the averages of the functions by which the topology is defined. In a sense, it is the set of such averages that is taken to constitute the definition of a measure. Course grainings go somewhat in this direction, as they correspond to using characteristic functions of the graining cells (the characteristic function of a set equals one on the set and zero elsewhere) as a description of the measure. This implies that one should be able to describe stationary nonequilibrium states without having to resort to the arbitrariness of such expedients. Indeed, the answer to the problem is in the relaxation of the requirement of looking at the phase space distributions directly, replacing it with the choice of the proper measure spaces and their topologies. In more physical terms, one must decide *a priori* what are the relevant observables and then study the evolution of the system in terms of the evolution of the averages of those variables.

Returning to the baker's map, we note that for most practical purposes it suffices to take powers of the phase variables as observables (x suffices in our case), given the density of polynomials in quite general function spaces. This corresponds to the very familiar technique, in nonequilibrium statistical mechanics, of studying the moments of phase space distributions rather than the distribution themselves. We see that the observables $\langle x \rangle$ has a well-defined limit as $M \to \infty$, whereas the Gibbs and Boltzmann entropies must be excluded from our function space because neither the distribution nor functions of it (like the logarithm) can be regarded as observables.

The observations presented in this section are quite important. Without coarse graining, a stationary distribution is never observed in this dissipative system, whether we start with a distribution on the *left lines* of the attractor or with a more general distribution in phase space. On the contrary, if we do a graining, stationarity emerges at a finite time determined by the size of the graining cells. Had we looked in more detail in the phase space, we would have found that the observed stationarity is indeed a *dynamical* one, that is not reproduced on all possible scales (thus it is not satisfactory from a theoretical viewpoint). It is better to furnish our distribution's space with the appropriate topology, defined by the set of observables that are relevant to the physics of the problem. In this way we see that it does take an infinite amount of time for the system to reach the stationary state, but then the stationary distribution, indirectly defined by the averages that it determines on the space of observables, is invariant under the map by definition. It would appear, then, that stationarity is an effect of our limited ability of observation and that what distinguishes equilibrium states from stationary nonequilibrium ones is that nonequilibrium distribution functions can always be observed on scales in which they appear nonstationary. In the case of course graining we decide that the space is granular to our measurement tools and the dynamics look different with different tools; in the other case, using a stronger or weaker topology (measuring more or fewer observables) has the same effect. On the contrary, equilibrium states such as the measure preserving baker's map appear stationary on all scales.

III. ORBITAL MEASURES

It has been proposed to use unstable periodic orbits (UPOs) to calculate a sequence of approximations to the stationary measure of a number of interesting dynamical systems. Several different but closely related approaches have been used to calculate averages of observables. There is the periodic orbit expansion (POE) method [7,14–16] and the cycle expansion method of Artuso, Aurell, and Cvitanovic [17]. These two methods are based upon the thermodynamic formalism of Ruelle and are equivalent in the limit of long periods, in that they converge to the same limiting measure. However, their convergence properties are in general different. In either of these approaches the average of a system property B is written in terms of weighted contributions from UPOs. In particular, for the POE method we can write

$$\langle B \rangle = \lim_{n\to\infty} \langle B \rangle_n = \lim_{n\to\infty} \frac{\sum_{i\in P_n} \Lambda_i^{-1} \int_0^{\tau_i} B(s)ds}{\sum_{i\in P_n} \tau_i \Lambda_i^{-1}} , \quad (5)$$

where P_n is set of UPOs of length n, Λ_i is the largest Lyapunov number (if there is only one expanding direction), and τ_i is the period of the ith UPO. The largest Lyapunov number is $\Lambda_i = \exp(\lambda_i \tau_i)$, where λ_i is the largest Lyapunov exponent. In Eq. (5) we have considered a system with continuous time evolution in which a Poincaré section can be introduced to transform the effective motion to the study of the mapping from one intersection with the Poincaré surface of section to a subsequent intersection. In this way an UPO of length n implies that on the nth intersection with the Poincaré surface the trajectory has returned its initial condition. In general, the time taken to evolve from the initial condition to the nth intersection will depend upon both the initial condition and the underlying (continuous time) dynamics. However, if the dynamical system is a mapping (for example, the baker's map), then $\tau_i = n$ for all periodic points of order n.

An argument for the weighting of periodic orbits by Λ_i^{-1} can be found in Ref. [10], where a general smooth mapping is considered, $\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n)$. In particular, the authors of Ref. [10] discuss the case of maps with hyperbolic attractors that have a dense set of periodic orbits. Their argument can be summarized as follows. Imagine that we partition the space into cells C_i, where each cell has as its boundaries stable and unstable manifolds. If the cells are very small, the curvature of the boundaries will be slight and can be neglected, so we can regard the cells as parallelograms. Consider a given cell C_k and a large number of initial conditions sprinkled within that cell according to the natural probability measure on the attractor. If we iterate each of these initial conditions n times, then a small fraction of the initial conditions may return to the cell C_k. Since we assume the attractor to be ergodic and mixing, this fraction is asymptotically equal to the natural measure on the attractor in the cell $\mu(C_k)$.

In Fig. 2 we construct the nth forward iterate of C_k and denote it by $+$ and the nth backward iterate and denote it by $-$. The intersection of $+$ and $-$ must contain a fixed point j of $\mathbf{F}^n$ and the magnitudes of its unstable and stable eigenvalues are Λ_{1j} and Λ_{2j}. The rectangle $-$ is such that all the points in $-$ return to C_k after n iterations. Denoting the lengths of the sides of the cell C_k by ξ_k and η_k, we see that the initial rectangle has area $\xi_k \eta_k \Lambda_{1j}^{-1}$ and the final rectangle has area $\xi_k \Lambda_{2j} \eta_k$. Since the dynamics is expanding in the vertical direction, the attractor measure is expected to vary smoothly in this direction (i.e., it is expected to be a SRB measure) [18]. As the cell is small, we can treat the measure on the attractor as if it were essentially uniform along the rectangle $-$. Thus the fraction of the measure of C_k occupied by the strip is Λ_{1j}^{-1}. Since for $n\to\infty$ the fraction of initial conditions starting in C_k that return to it is the unnormalized density $\mu(C_k)$, we have

$$\mu(C_k) = \lim_{n\to\infty} \left[\sum_{\left\{\substack{\text{period-}n \\ \text{orbits in } C_k}\right\}} \Lambda_{1j}^{-1} \right] . \quad (6)$$

Also note that, as n gets larger, Λ_{1j}^{-1} and Λ_{2j} get exponentially smaller while the number of fixed points of $\mathbf{F}^n$ in C_k grows exponentially. Since we imagine that we can make the partition into cells as small as we wish (with reasonably smooth boundaries) by a covering of cells, the result in Eq. (6) follows. Note that for maps, the period of the periodic orbit does not appear in the formula for the measure, as at every level n all the orbits have exactly the same period, which can be factored out through the normalization constant. On the contrary, for flows the periods of different orbits, which account for the longitudinal direction of the flow, are different and cannot be simplified through the normalization constant. The generalization of this argument to systems with Poincaré sections of more than two dimensions leads to a stability weight that is the product of the Lyapunov numbers for all expanding directions of the dynamics. In all cases the measure defined in Eq. (6) must be explicitly normalized in order to calculate averages.

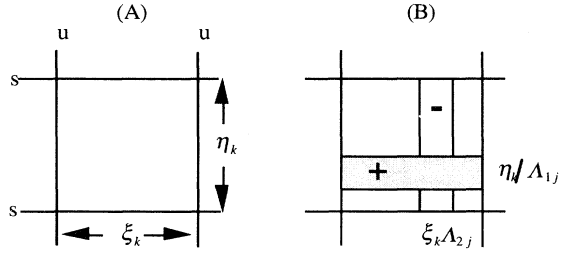

FIG. 2. (a) Rectangle formed by the intersections of a pair of stable and a pair of unstable manifolds. In (b) the rectangle marked $+$ is that part of the nth forward iterate of the initial rectangle in (a) that returns to C_k. Similarly, the rectangle marked $-$ is that part of the nth backward iterate of the initial rectangle (a) that returns to C_k. Clearly, as the intersection of $+$ and $-$ is wholly contained within the initial rectangle, the intersection contains an nth-order hyperbolic periodic point.

The cycle expansion formula can be derived through the definition of a Ruelle ζ function whose zeros are related to the average of the phase variable B as follows (see Ref. [17] for details). For simplicity, take a system with complete, binary symbolic dynamics (such as the baker's map); the relevant ζ function takes the form

$$\zeta(\beta,Q)^{-1} = 1 - t_0 - t_1 - [t_{01} - t_0 t_1] - [t_{001} - t_0 t_{01}] - [t_{011} - t_{01} t_1] - \cdots , \quad (7)$$

where

$$t_i = \frac{e^{\beta B^n(x_{i0}) - nQ}}{\Lambda_i} , \quad (8)$$

$B^n(x_{i0})$ is the sum of the values of the phase variable B along the periodic orbit beginning from x_{i0}. Differentiating $\zeta(\beta,Q)^{-1} = 0$ with respect to β, the average of B is given by

$$\langle B \rangle = \left. \frac{dQ}{d\beta} \right|_{\beta=0} . \tag{9}$$

The terms in square brackets in Eq. (7) are called *curvature corrections* and, for particular systems and particular phase variables B, they are precisely equal to zero, thus making the formalism very simple and effective.

The two methods outlined here—periodic orbit expansions and cycle expansions—produce the same averages in the limit where the lengths of the orbits tend to infinity, thus they are equivalent from a theoretical point of view. However, their performance in practical calculations, which are done at finite period, is different, as they arrange in a slightly different fashion the same set of data. Depending upon the circumstances, periodic orbit expansions may converge faster than cycle expansions (e.g., in the equilibrium Lorentz gas [15]) or vice versa (e.g., for the baker's map in the next section).

IV. ORBITAL MEASURES FOR THE BAKER'S MAP

As a simple example of the application of periodic orbit theory, let us consider the generalized baker's map. Observe that it is possible to label the orbits by assigning a symbol to each of the two pieces of the map; for instance, we can use the symbol 0 to identify a point in the region $0 \le y < \frac{1}{2}$, so that the next step in its evolution is determined T_0, and we can assign the symbol 1 to a point in $\frac{1}{2} \le y < 1$, which is evolved by T_1. These are

$$(x,y) \xrightarrow{T_0} \left[\frac{x}{3}, 2y \right], \quad (x,y) \xrightarrow{T_1} \left[\frac{x}{3} + \frac{1}{2}, 2y - 1 \right] . \tag{10}$$

With the aid of this symbolic dynamics it is easy to find all the periodic orbits for the baker's map and then to visualize how their points cluster in the support of the measure. This is quite a trivial exercise for the baker's map, as its Lyapunov exponents are uniform in space and the expansion and contraction rates in phase space do not change in time.

The periodic orbits for the baker's map up to period four can be easily listed in terms of our binary symbolic dynamics, given in Table II.

TABLE II. Cycle points for the perioidc orbits of lengths 1–4.

Length	Symbol	Cycle points
1	{0}	(0,0)
	{1}	$(\frac{3}{4},1)$
2	{01}	$\{(\frac{3}{16},\frac{2}{3}),(\frac{9}{16},\frac{1}{3})\}$
3	{001}	$\{(\frac{3}{52},\frac{4}{7}),(\frac{27}{52},\frac{1}{7}),(\frac{9}{52},\frac{2}{7})\}$
	{011}	$\{(\frac{3}{13},\frac{6}{7}),(\frac{15}{26},\frac{5}{7}),(\frac{9}{13},\frac{3}{7})\}$
4	{0001}	$\{(\frac{3}{160},\frac{8}{15}),(\frac{81}{160},\frac{1}{15}),(\frac{27}{160},\frac{2}{15}),(\frac{9}{160},\frac{4}{15})\}$
	{0011}	$\{(\frac{3}{40},\frac{4}{5}),(\frac{21}{40},\frac{3}{5}),(\frac{27}{40},\frac{1}{5}),(\frac{9}{40},\frac{2}{5})\}$
	{0111}	$\{(\frac{39}{160},\frac{14}{15}),(\frac{93}{160},\frac{13}{15}),(\frac{111}{160},\frac{11}{15}),(\frac{117}{160},\frac{7}{15})\}$

We see that at every period n there is a one-to-one correspondence between rectangles and the fixed points of the n times iterated map. Each rectangle of nonvanishing probability contains one fixed point of the nth iterate of the uniform distribution. Indeed, due to the smoothness of the map (except at $y=\frac{1}{2}$), we have that if a point of a periodic orbit is in the interior of such a band, it remains in the interior of a band of nonzero probability for all the iterates of the map.

So now let us compare the results on the limit of the transient measures, given in Sec. II, with those we get from the limit of orbital measures. In the case where the phase variable is $B \equiv x$, all the curvature correction terms in the Ruelle ζ function vanish and we have $[\zeta(\beta,Q)]^{-1} = 1 - t_0 - t_1 = 0$ from which $\langle x \rangle = \frac{3}{8}$ follows immediately. Similarly, the distribution of periodic orbits in phase space is such that the periodic orbit expansion gives the correct result from the smallest period $n=1$. However, for $B \equiv x^2$ the curvature terms are nonzero and we need the full expression for the ζ function, as well as we need to extrapolate the results of POE to infinite periods. A comparison of the different methods is presented in Table III.

Given this picture, we can compute the asymptotic behavior of the distribution function, in its approach to the stationary state, by making use of our knowledge of unstable periodic orbits. First we have, asymptotically, that the measure of subsets of phase space that do not contain a periodic orbit is vanishing. Second, in a neighborhood of a given periodic orbit, phase space volumes change in one iteration according to the formula $V \to \Lambda_1 \Lambda_2 V$, where Λ_i is the ith Lyapunov number of the orbit, that is to say, the ith eigenvalue of the absolute value of the Jacobian matrix of the map along the orbit. Consequently, the distribution function changes as $f \to \Lambda_1^{-1} \Lambda_2^{-1} f$, in such a neighborhood. In our case $\Lambda_1 \Lambda_2 = \frac{2}{3}$ for all orbits, so we get that the distribution function grows a factor $\frac{3}{2}$ at every iteration in all the regions of the attractor. This is trivially correct, as we saw in Sec. II; however, the use of periodic orbits gives also the regions in which the stationary state is concentrated. With this in mind, we now turn to the nonequilibrium Lorentz gas.

TABLE III. Comparison of the results obtained for $\langle x^2 \rangle$ from the transient measures (rectangles), the periodic orbit expansion (POE), and the cycle expansion. The values for the POE and cycle expansion at finite length bear no relation to the transient measure, but are simply successive approximations to the stationary average.

Length	Rectangles	POE	Cycles
0	0.333 333 3		
1	0.245 370 4	0.281 250 0	0.281 250 0
2	0.221 707 8	0.175 781 3	0.228 515 6
3	0.214 449 0	0.194 711 5	0.216 346 2
4	0.212 099 0	0.207 421 9	0.212 695 3
5	0.211 323 8	0.206 869 8	0.211 518 6
6	0.211 066 2	0.211 667 2	0.211 130 7
	0.210 937 5		

V. THE LIOUVILLE EQUATION AND KAWASAKI DISTRIBUTION FUNCTION

The classical Liouville equation is obtained by considering an infinitesimal element of phase space containing a fixed number of ensemble members [1]. The distribution function (or density of the associated measure) is then defined to be the number of ensemble members divided by the volume of the phase element, apart from a normalization constant. The rate of change of ensemble members is calculated by looking at the flux through the surface of the element. Taking the infinitesimal limit, with some assumptions about the existence of a first derivative of the density, the Liouville equation follows:

$$\frac{d}{dt}f(\Gamma)=-f(\Gamma)\frac{\partial}{\partial T}\dot{\Gamma}\ , \tag{11}$$

where $\Gamma\equiv(q_1,\ldots,q_{3N},p_1,\ldots,p_{3N})$ is the phase space point representing the full description of the state of the system.

An important point to note here is that the Liouville equation constructs a complete balance between the loss and gain of ensemble members in each phase space volume element. As a result, the total number of ensemble members is constant and we expect the distribution function to remain normalized. This is quite different from the derivation of the weights for orbital measures in Sec. III, where we consider a cell whose boundaries are the stable and unstable manifolds, and we are only interested in those initial conditions that return to the cell. We disregard the initial conditions that leave the cell. If no initial conditions return to the cell then the measure of the cell is zero. Another important distinction between the Liouville equation and orbital measures is that the Liouville equation requires an initial distribution and calculates the time evolution of that distribution. Therefore, the Liouville equation correctly describes the behavior of the *transient* measures. The orbital measures, instead, are a sequence of approximations to the long time limit of the transient ones, which we term the stationary state. The orbital measures are determined by the dynamics alone and the only restriction on the initial distribution is that it is taken from the set of those that converge to the ergodic attractor approximated by the given set of periodic orbits. Thus any information about the initial measure is lost in the POE approach for most of the practical circumstances (e.g., in the case of Ref. [19], all absolutely continuous measures with respect to Liouville measure evolve to a unique multifractal ergodic attractor). As a consequence, the Kawasaki distribution on the attractor, which is approximated by orbital measures of larger and larger period, is of a different nature to the transient, which has support on the whole phase space. Indeed, the transient retains the full dimension of the phase space, at all finite times, while the stationary measure has a lower (fractal) dimension. The reason is that the stationary state is an idealized state, where it is assumed that an infinite time has elapsed since the initial preparation of the ensemble. This produces a support that has vanishing Liouville measure, because the contracting effects of the dynamics reduce the transverse thickness of the support to zero. Therefore, in the stationary state the contracting directions play no role, while both contracting and expanding directions are important in the transient. Despite these differences, the transient distribution approaches the ideal stationary state as $t\rightarrow\infty$, until the two are indistinguishable to our observations. This is why we can give physical meaning to the stationary states, for which a number of mathematical techniques have been developed, but are not available for the study of transients.

The derivation of the Kawasaki formalism for the evolution of the distribution function [20,21] is a formally exact consequence of the Liouville equation (given that the distribution function is once differentiable). For a system whose equations of motion can be written as

$$\dot{\mathbf{q}}_i=\frac{\mathbf{p}_i}{m}+\mathbf{C}_i\cdot\mathbf{F}_e,\quad \dot{\mathbf{p}}_i=\mathbf{F}_i+\mathbf{D}_i\cdot\mathbf{F}_e-\alpha\mathbf{p}_i\ , \tag{12}$$

where $\mathbf{C}_i(\Gamma)$ and $\mathbf{D}_i(\Gamma)$ are the phase variable couplings to the external field $\mathbf{F}_e$ (the rank of $\mathbf{C}_i$ and $\mathbf{D}_i$ is such that when either is contracted into the external field $\mathbf{F}_e$ the result is a vector). The Kawasaki formula for the distribution function can be expressed as

$$f(\Gamma,t)=f(\Gamma,0)\exp\left[-\beta F_e\int_0^t J(\Gamma(-s))ds\right]\ , \tag{13}$$

where f is the distribution function, which depends on the phase space point Γ and on the time t, while $J(\Gamma)$ is the dissipative flux. The introduction of an external field that supplies energy to the system requires a thermostating mechanism to obtain stationarity (here we use the Gaussian thermostat). $J(\Gamma)$ is then defined through the rate of change of internal energy H_0

$$\begin{aligned}\dot{H}_0&=\sum_{i=1}^{N}\left[\frac{1}{m}p_i\cdot\dot{p}_i-F_i\cdot\dot{q}_i\right]\\&=-\sum_{i=1}^{N}\left[-\frac{1}{m}p_i\cdot D_i+F_i\cdot C_i\right]F_e-3NkT\alpha\\&=-J(\Gamma)F_e-3NkT\alpha\ .\end{aligned} \tag{14}$$

From the form of Eq. (13) we have two possibilities. We can consider finite values of t, and study the evolution of the distribution function with time (that is, study the transient behavior), or we can take the limit $t\rightarrow\infty$, and consider the steady state distribution function. However, there are two major difficulties with the Kawasaki formalism. First, it is easy to see that for physically relevant trajectories, the exponent in Eq. (13) is positive, which implies that the probability of the initial conditions for these trajectories $f(\Gamma_0,t)$ diverges to infinity as $t\rightarrow\infty$. Second, the Gibbs entropy, defined as in Sec. II, diverges to negative infinity, linearly with time [22]. This is analogous to the divergence of the distribution, and therefore of entropy, on the set of left lines that we observed in the baker's map. Despite the fact that the Kawasaki formula follows immediately from the Liouville equation, these two difficulties have remained unresolved and at times have even led to doubts about the Kawasaki formalism. The doubts are also raised because stationary states for

systems of the kind described here do not have a density in phase space, as the physical measure is singular and concentrated on a fractal subset of the phase space. We will see that these difficulties do not constitute a problem if the transient and the stationary state are treated in different fashions.

VI. KAWASAKI DISTRIBUTION FOR THE LORENTZ GAS

The particular example of the nonequilibrium Lorentz gas that we consider here has one pointlike particle moving in an arbitrarily complex periodic lattice of general convex scatterers, with an external field in the x direction and a Gaussian thermostat. The equations of motion are

$$\dot{\mathbf{q}}=\frac{\mathbf{p}}{m},\quad \dot{\mathbf{p}}=\mathbf{F}+\mathbf{F}_e-\alpha\mathbf{p} \tag{15}$$

and, given the symmetry of the model, we can reduce the study of its dynamics to an elementary cell (EC), whose replications tile the whole plane. Thus we switch from the original problem in the plane to an equivalent one in the EC. In this way, the fact that a particle leaves the EC in the plane from one of its sides and a new one enters from the opposite side is represented in the reduced problem by the motion of a unique particle that keeps going round a bounded surface, similarly to the motion on a torus. Obviously, we must be careful while transforming one problem into the other, so that appropriate correspondences are made between the phase variables in the plane and those that represent them in the EC. For instance, in the EC there is no diffusion of particles because all trajectories remain bounded. However, one can compute the relevant quantity for diffusion in the plane even in the EC by looking at the total distance *traveled* by a particle rather than just at the distance between initial and final points along the trajectory. This distinction between the distance traveled and the distance between points is important in the EC, while it is not so in the plane.

Once the Lorentz gas problem has been transformed to the EC, many tools from the theory of dynamical systems and chaos become available, in particular, the POE technique. Quite clearly, the Lorentz gas with hard core scatterers is not smooth, so the POE theory does not strictly apply. Nonetheless, the method has been shown to work well both with and without an external field [7,9,12,14,15,23,24] and a possible mathematical explanation of this can be found in [7,25]. We now explore the connections between the POE and Kawasaki formalisms for this system. The dissipation function for the field-dependent Lorentz gas is given by $J=p_x/m=\dot{x}$. Substituting this into Eq. (13) gives the time evolution of the distribution f according to the Kawasaki formula:

$$\begin{aligned} f(\Gamma,t)&=f(\Gamma,0)\exp\left[-\beta F_e\int_0^t\dot{x}(\Gamma,-s)ds\right]\\ &=f(\Gamma,0)e^{\beta F_e\Delta x(\Gamma,-t)}. \end{aligned} \tag{16}$$

It is most important to realize that Eqs. (13) and (16) relate the density in the initial distribution at position Γ to the density of the *same* phase space position some time later. Moreover, this relation can be derived with the same form on both phase spaces: the plane and the EC. In the special case where Γ belongs to a periodic orbit of period τ, we can apply the Lyapunov sum rule [7] $\beta F_e l(\tau)=(\lambda_1+\lambda_2)\tau$. This rule relates the total deformation of the phase space around a trajectory, in a time τ, to the displacement along the trajectory in the direction opposite to the external field $l(\tau)$. For this, it is irrelevant whether we are in the plane or in the elementary cell. In fact, going down that trajectory in the plane for multiples of τ, the same pattern is reproduced all times and so the Lyapunov numbers computed for the periodic orbit are also the correct ones for that trajectory. Then, in the plane we have $l(\tau)=\Delta x(\tau)$ and $\Delta x(-t)=-\Delta x(t)$, from which, substituting in Eq. (16), we obtain

$$\begin{aligned} f(\Gamma,\tau)&=f(\Gamma,0)e^{-\beta F_e\Delta x(\tau)}\\ &=f(\Gamma,0)e^{-(\lambda_1+\lambda_2)\tau}=f(\Gamma,0)\Lambda_1^{-1}\Lambda_2^{-1}, \end{aligned} \tag{17}$$

where Λ_1 and Λ_2, respectively, are the smallest and largest Lyapunov numbers for the particular trajectory. This shows another example where care must be used in going from the plane to the EC. Relation (17) does not derive from Eq. (16) for a periodic orbit and so it does not apply directly to periodic orbits in the EC, because there $l(\tau)\neq\Delta x(\tau)=0$. Nonetheless, it is the correct expression for the variation of the distribution around a periodic point of a generic dynamical system and Eq. (17) constitutes a first link between the Kawasaki formalism and the results of periodic orbit theory. Then, as the relation is correct around the periodic orbit in the elementary cell, its unfolding remains valid in the plane and we have a relation valid both for the full and the reduced Lorentz gas problems. The reason is that the Kawasaki formula correctly gives the evolution of the distribution during the transient in the plane, which happens to be identical to the corresponding evolution in the elementary cell.

In fact, it is not necessary to begin with the Kawasaki formalism to obtain this result. We can also obtain it from more elementary considerations, as done for the baker's map. In the neighborhood of a periodic point we see that the change in the volume of the element is determined by the Lyapunov numbers and is given by $V(t)=V_0\Lambda_1\Lambda_2=V_0e^{-(\lambda_1+\lambda_2)t}$. Therefore, if the element is small enough and the distribution on it is smooth enough, we have that the distribution changes in time as the inverse of the change in volume giving Eq. (17). If we continue this argument further by traversing the periodic orbit repeatedly we find that each traversal gives another factor of $\Lambda_1^{-1}\Lambda_2^{-1}$ and, because the contraction is stronger than the expansion, the density grows indefinitely. The same argument applies to each periodic orbit in the EC and the result is that the distribution grows without bounds around UPOs with nonvanishing displacement in the plane, while it goes to zero on the parts of phase space that have no orbits.

Now, it is not clear from the Kawasaki formula whether or not the distribution is normalized for any time after $t=0$. This is typical of the apparent behavior of

Kawasaki distribution functions. On physical grounds the distribution function must be normalized because ensemble members are neither created nor destroyed, but the derivations do not show this clearly. Indeed, at every point where there is a definite stationary current the distribution seems to blow up, despite the fact that the point may belong to the support of the stationary state. However, looking at how the stationary state is approached from the transient evolution, we realize that there must be a relation between the reduction in support and the growth of singularities, which results in a normalized distribution, similar to the baker's map.

Note that all these considerations apply in the transient, not in the stationary, state; however the Lyapunov numbers of the periodic orbits are characteristic of the stationary state. Thus the results derived above are to be intended as valid asymptotically, that is to say, in the approach to the stationary state and very close to it.

The usual Schrödinger representation of the Kawaskai formula is not the only possible expression. Evans and Searles [26] have obtained a form that is suitable for Lagrangian phase space elements

$$f(\Gamma(t),t)=\exp\left[\int_0^t 3N\alpha(\Gamma(s))ds\right]f(\Gamma(0),0)\ , \qquad (18)$$

which we can see involves integrating α along a trajectory. The weakness of this approach is that, in general, the integral of α is not an invariant quantity unless the initial point corresponds to a periodic orbit. If the initial point is a periodic orbit then we would obtain the product of Lyapunov numbers as in Eq. (17).

VII. IS THE KAWASAKI DISTRIBUTION STATIONARY?

We consider the reduction of the Lorentz gas to the elementary cell. Then we can study the problem of stationarity of the Kawasaki distribution in the stationary state from two points of view involving the use of periodic orbits. The first is to write the Liouville equation on a periodic orbit, and then look at the evolution of a given distribution on such periodic orbits, in terms of the solution of the Liouville equation. This can be done by noting that, when the phase space is just one line in the plane, we can parametrize the points on that line by a unique coordinate q, representing the distance traveled from a fixed origin, and then solve the Liouville equation for $f(q,\dot{q})$. Clearly, a uniform measure (in configuration space) over a periodic orbit with constant speed must be stationary, as time evolution along it merely corresponds to a permutation of points, and the total distribution of points is unchanged. Thus, on periodic orbits we easily get stationarity of the distribution from the Liouville equation. Note that a periodic orbit is the only trajectory to which we can assign a uniform density of particles, because the other trajectories would end up with an unnormalized distribution. This shows that, pointwise, a distribution on a generic chaotic trajectory cannot be stationary, not even in the stationary state, similarly to the results for the baker's map.

The second way to prove stationarity of distributions on periodic orbits provides also an independent check of the Kawasaki formalism in this context. A uniform distribution on the orbit may be interpreted as having a number N of particles, equally spaced along its length, and then we have

$$\begin{aligned} f(\Gamma,t)&=f(\Gamma,0)\exp\left[-\beta F_e\int_0^t-\sum_i \dot{q}_{xi}(-s)ds\right] \\ &=f(\Gamma,0)\exp\left[\beta F_e\sum_i \Delta q_{xi}(-t)\right]\ . \end{aligned} \qquad (19)$$

Now, our distribution is stationary if $\sum_{i=1}^{N}\Delta q_x(-t)=0$ for every instant of time t, but we know that this is not the case except for all times that are a multiple of τ/N, where τ is the period of the orbit. Indeed, in such a time all the particles move from their current site to the nearest one, in the direction of motion, i.e., each one of them moves of an amount τ/N in the cycle so that the total displacement vanishes. Then, taking the limit $N\to\infty$, we obtain that periodic orbit distributions are stationary also according to the Kawasaki formalism.

Once the stationarity of the distributions on periodic orbits has been established (one way or another), its most important consequence is that the Kawasaki distribution, supported on the attractor, $\mu_t^{(\infty)}$ say, can also be proved to be stationary (in the appropriate weak topology). Here the subscript t indicates the possibility that the distribution changes with time. Indeed, if B is an observable taken from the set that defines the topology on the space of measures under investigation and $\mu_t^{(\tau)}$ is the orbital measure of period τ at time t we have

$$\begin{aligned} \frac{d}{dt}\langle B\rangle&=\frac{d}{dt}\int B(\Gamma)d\mu_t^{(\infty)}(\Gamma) \\ &=\frac{d}{dt}\lim_{\tau\to\infty}\int B(\Gamma)d\mu_t^{(\tau)}(\Gamma)=0\ , \end{aligned} \qquad (20)$$

which is the statement of stationarity of $\mu_t^{(\infty)}$ in the chosen topology, if $\mu_t^{(\infty)}$ is the limit of the orbital measures. In particular, let $B=1$ and consider the Kawasaki propagator $\exp(\beta F_e\sum\Delta q_{xi}(-T))$ as a phase variable for fixed parameter T. By the POE theory we get

$$\begin{aligned} 1&=\lim_{\tau\to\infty}\int d\mu_T^{(\tau)}(\Gamma) \\ &=\lim_{\tau\to\infty}\int\exp\left[\beta F_e\sum_{i=1}^{N}\Delta q_{xi}(\Gamma,-T)\right]d\mu_0^{(\tau)}(\Gamma) \\ &=\int\exp\left[\beta F_e\sum_{i=1}^{N}\Delta q_{xi}(\Gamma,-T)\right]d\mu_0^{(\infty)}(\Gamma) \\ &\equiv\int d\mu_T^{(\infty)}(\Gamma) \end{aligned} \qquad (21)$$

if the orbital measures are normalized to start. Here the first two integrals are equal by definition of Kawasaki distribution on periodic orbits. Thus the distribution function obtained from Kawasaki evolution is always normalized for stationary states. Note that the orbital measures and the fractal measure have disjoint support, thus the equality between the second integral and the third integral of Eq. (21) is not trivially true, as the exponentials

in the integrals are evaluated over disjoint sets. Such an equality relies on the applicability of POE techniques. The last equality, instead, is just the definition of the evolved stationary measure at time T, in the assumption that the Kawasaki propagator can be used in this context. The argument is that we assume the stationary state to have a density along the unstable manifold of the attractor, similarly to what is done in Ref. [27]. Then, the Liouville equation and the Kawasaki formalism can be derived through balances in the volume elements of the attractor in the same way as they are derived in the full phase space for the transient measures. Under the same hypotheses, we get the result of Eq. (20) for the Kawasaki evolution

$$\int B(\Gamma)\exp\left[\beta F_e \sum_{i=1}^{N} \Delta q_{xi}(\Gamma,-T)\right] d\mu_0^{(\infty)}(\Gamma)$$
$$= \lim_{\tau\to\infty} \int B(\Gamma)\exp\left[\beta F_e \sum_{i=1}^{N} \Delta q_{xi}(\Gamma,-T)\right] d\mu_0^{(\tau)}(\Gamma)$$
$$= \lim_{\tau\to\infty} \int B(\Gamma)1 d\mu_0^{(\tau)}(\Gamma) = \int B(\Gamma) d\mu_0^{(\infty)}(\Gamma) \qquad (22)$$

for all times $T=\tau/N$ (for instance, we can take $T<\varepsilon$, arbitrarily small, and then take the limit $N,\tau\to\infty$ in order to keep their ratio fixed or to make it progressively smaller). Again, we see that the stationarity of the Kawasaki distribution function, in the chosen topology, follows from POE theory and from the stationarity of the distribution on orbital measures.

VIII. CONCLUSION

In this work we have outlined some of the distinguishing features of nonequilibrium steady states for dissipative systems with reversible microscopic dynamics, as opposed to equilibrium stationary states. The baker's map has been used in this context as a pedagogical example, which exhibits many qualitative similarities with more realistic models. Among these similarities we have found the divergence of the entropy and of the density distribution in the approach to the stationary state. Moreover, for dissipative systems we have seen that the stationary distribution is not directly accessible, thus requiring the use of appropriate topologies for its description, while this was not necessary for the measure preserving cases. We have also tested periodic orbit theory as a tool for the computation of thermodynamic quantities and we have compared it to the method based on the extrapolation of transient measure results, which is equivalent to a method proposed in the molecular dynamics literature [28].

The nonequilibrium Lorentz gas has been discussed as an example of a "more realistic" model and we have seen that the Kawasaki formalism applies to the transient behavior of the phase space distribution. This evolves according to the contraction and expansion rates of the stable and unstable manifolds of the system, at all finite times. Moreover, for large times the evolution is guided by the unstable periodic orbits, whose stability eigenvalues determine the rates of change of phase space volumes around them. In contrast, in the stationary state the contracting directions play no role, because the relevant measure is concentrated on the unstable manifold, which we assume to be transverse to the stable manifold (hyperbolicity assumption [27]). We have also assumed that the stationary measure, which vanishes in regions with no periodic orbits, is sufficiently well behaved (SRB-like) on the surface elements of the unstable manifold. Therefore, we have applied the Kawasaki formalism to the stationary state and we have investigated its properties in terms of periodic orbit theory. We conclude that there are no inconsistencies in the treatment of stationary states using Kawasaki formulas, such as the loss of normalization along the evolution, at least in those cases where our hypotheses are verified. Furthermore, we argue that the nonequilibrium Lorentz gas is one such case, because periodic orbit theory has proved applicable and effective in the study of its properties [7,9,14].

To decide how far an ensemble is from equilibrium we need some indicator function. All the information to do this should be contained within the measure, but we have had to exclude functions of the distribution such as the Gibbs and Boltzmann entropies, as they are not observables. However, there is a class of functions of the distribution that can be included as observables and these are the set of generalized dimensions $D(q)$ [29]. The difference between these functions and the ones we exclude is that these are based upon an initial coarse graining of the measure. A finite result emerges in the limit as the size of the graining goes to zero. These functions have the properties that we require of a nonequilibrium entropy: they are maximum at equilibrium and uniformly decreasing as we move away from equilibrium. As such we might expect to be able build a thermodynamic description based on an entropylike function of this type. The problem is that, in general (for multifractals), there is an infinite class of such functions, parametrized by the real variable q. For the baker's map considered here (which is a uniform fractal) $D(q)=1+\ln 2/\ln 3$, where the 3 in the denominator comes from the contraction rate of $\frac{1}{3}$. Considering a variable contraction rate of $1/\alpha$, we see that the generalized dimension would be $D(q)=1+\ln 2/\ln\alpha$. This generalized dimension is equal to the full space dimension when the mapping is area preserving and decreases monotonically as α increases from the (area preserving) value of 2. The interesting fact is that generalized dimensions can be calculated from periodic orbits. In fact, as the periods of the orbits increase, each orbit contributes essentially the same amount to the measure and the dimension is a measure of the average local clustering of periodic points.

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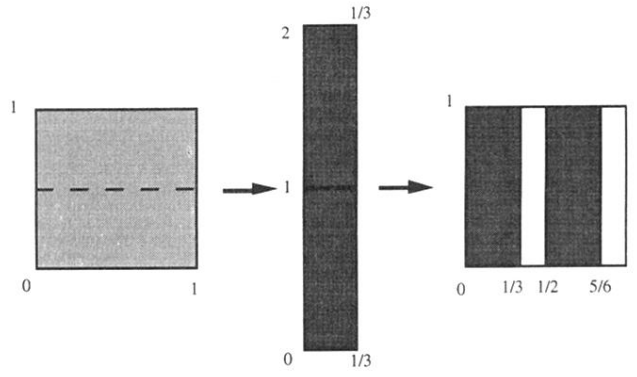
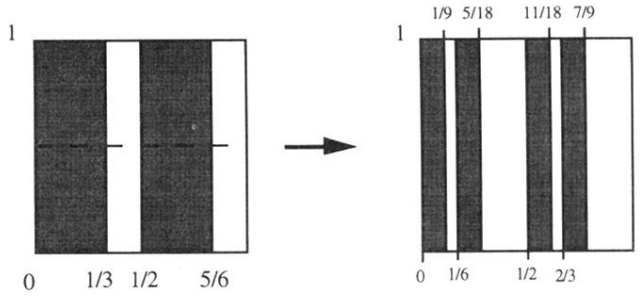


(a)



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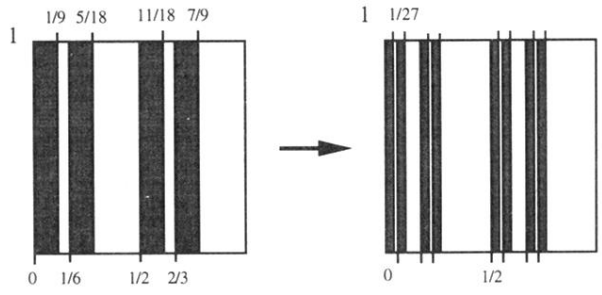

(c)

FIG. 1. (a) First iteration of the baker's map acting on a uniform measure on the square. (b) Second iteration. (c) Third iteration. At each step the number of rectangles increases by a factor of 2 and the density on each increases.